Exam 1: Solutions

1. State whether the statement is TRUE or FALSE. Justify your answer. In addition, define/briefly explain the underlined concepts.

   (a) A pure action is rationalizable if it survives the iterative elimination of strictly dominated strategies. (You may assume two players if you wish.)

   Solution. TRUE. Rationalizability (see below) and iterative elimination of strictly dominated actions are equivalent. (With more than two players, this holds only if we allow correlated beliefs.)

   An action is rationalizable if it survives the iterative elimination of never-best responses.

   (b) In the following game of incomplete information, the unique Bayes Nash equilibrium is that P1 plays U if $\theta_1 = 1$ and D if $\theta_1 = -1$ and P2 plays L.

   Players' actions and payoffs are (P1 is the row player and P2 is the column player):

	L	R
U	$\theta_1, 1$	$0, 0$
D	$0, 0$	$1, 1$

   where θ_1 is either -1 or 1. P1 knows θ_1 but P2 does not. The probability that θ_1 is 1 is $p > 0.5$.

   Solution. FALSE. (D, R) is also a BNE.

   In games of incomplete information some players do not know the game structure, e.g. payoffs that follow from an action profile. (Not to be confused with imperfect information, which means that players do not observe all actions taken before them.)

   (c) A Sequential Equilibrium need not be a weak Perfect Bayesian Equilibrium if the game has incomplete information.

   Solution. FALSE. The difference between the two solution concepts is that SE puts more restrictions on off-path beliefs than wPBE and hence every SE is necessarily wPBE.

   A weak Perfect Bayesian equilibrium is a pair of strategy profiles and beliefs such that the strategies are sequentially rational in every information set given the beliefs and the beliefs are driven from the strategy profile by Bayes rule in every information set reached with a positive probability.

2. Two firms simultaneously decide whether to enter a market. Firm i's entry cost c_i is $\overline{c}$ with probability p and $\underline{c}$ with probability $1-p$. Each c_i is private information to Firm i. Firm i has payoff $\Pi^m - c_i$ if i is the only firm to enter and $\Pi^d - c_i$ if both firms enter. Not entering yields a payoff 0. Assume that $\Pi^m > \overline{c} > \Pi^d > \underline{c} > 0$.

   (a) Formulate the game as a Bayesian game.

   Solution.

   - Players: F1 and F2
   - Types: $c_1 \in \{\underline{c}, \overline{c}\}$ and $c_2 \in \{\underline{c}, \overline{c}\}$
   - Type distribution: $Pr(c_i = \overline{c}) = p$ for both $i \in \{1, 2\}$
   - Actions and payoffs:

	E	N
E	$\Pi^d - c_1, \Pi^d - c_2$	$\Pi^m - c_1, 0$
N	$0, \Pi^m - c_2$	$0, 0$

(b) For what values of p there is a symmetric Bayes Nash equilibrium where both players enter if and only if their cost is $\underline{c}$?

Solution.
Assume F2 follows the suggested strategy. Payoff for F1 if E:

$$(1-p)(\Pi^d - c_1) + p(\Pi^m - c_1)$$

There is no profitable deviations from the suggested strategy if the above expected payoff is weakly positive when $c_1 = \underline{c}$ and weakly negative when $c_1 = \overline{c}$. This gives:

$$(1-p)(\Pi^d - \underline{c}) + p(\Pi^m - \underline{c}) \geq 0 \quad \text{holds always },$$
$$(1-p)(\Pi^d - c) + p(\Pi^m - \overline{c}) \leq 0 \iff p \leq \frac{\overline{c} - \Pi^d}{\Pi^m - \Pi^d}.$$

We can conclude that the suggested profile is a BNE if $p \in \left[0, \frac{\overline{c}-\Pi^d}{\Pi^m-\Pi^d}\right]$.

(c) Suppose $p\Pi^m + (1-p)\Pi^d > \overline{c}$. Find all Bayes Nash equilibria of the game.

Solution. The assumption $p\Pi^m + (1-p)\Pi^d > \overline{c}$ implies that a high cost firm is better off by entering if the other firm does not enter when it has high cost. Therefore, the strategy profile in part (b) is not a BNE in this case.
We also have $\overline{c} > \Pi^d$ and hence it cannot be that both types of both firms enter. Instead, we need to look at asymmetric equilibria and equilibria in mixed strategies.
Because $\Pi^d > \underline{c}$, low cost firms always enter. Hence, we can focus on what high cost firms do. Let α_i be the probability that high cost Fi enters. Now, high cost Fj gets the following payoff by entering:

$$((1-p) + p\alpha_i)(\Pi^d - \overline{c}) + p(1-\alpha_i)(\Pi^m - \overline{c}) = \Pi^d - \overline{c} + p(1-\alpha_i)(\Pi^m - \Pi^d).$$

Notice first that it is not possible that both firms are indifferent if $\alpha_1 \neq \alpha_2$ because the expression above is strictly decreasing in α_i. Hence, either $\alpha_1 = \alpha_2 \in (0,1)$ or $\alpha_i \in \{0,1\}$ for some i.
Let's start with $\alpha_1 = \alpha_2 =: \alpha \in (0,1)$: firms are indifferent if

$$\Pi^d - \overline{c} + p(1-\alpha)(\Pi^m - \Pi^d) = 0 \iff \alpha = \frac{p\Pi^m + (1-p)\Pi^d - \overline{c}}{p(\Pi^m - \Pi^d)}.$$

Next, suppose that $\alpha_2 = 0$. Then F1 wants to always play E. Is $\alpha_2 = 0$ a best reply to $\alpha_1 = 1$? Yes: entry by F2 yields $\Pi^d - \overline{c} < 0$ for F2 when $c_2 = \overline{c}$.
We have three BNEa. In all of them players play E if their cost is $\underline{c}$. They differ in what players do if their cost is $\underline{c}$: 1) symmetric mixing so that a player plays E w.p. α where $\alpha = \frac{p\Pi^m+(1-p)\Pi^d-\overline{c}}{p(\Pi^m-\Pi^d)}$; 2) Player 1 plays E and Player 2 plays N; 3) Player 1 plays N and Player 2 plays E.

(d) Now keeping the assumption that $p\Pi^m + (1-p)\Pi^d > \overline{c}$, suppose that Firm 1 enters first and Firm 2 decides about entry after observing Firm 1's decision. Find all weak Perfect Bayesian equilibria.

Solution. F2 enters if $c_2 = \underline{c}$ or if F1 has not entered. F1 understands this and enters always. Hence, the equilibrium is: F1 enters for both c_1, F2 enters if $c_2 = \underline{c}$ or if F1 has not entered; F2 assigns probability $1-p$ for $c_1 = \underline{c}$ if F1 enters and the belief can be anything if F1 does not enter.
Notice that beliefs do not affect optimal actions but we have defined wPBE so that beliefs are part of it. Also answers without beliefs will be accepted.

3. Consider the following stage game G (P1 is the row player and P2 is the column player):

	L	R
T	$1, -2$	$3, -2$
M	$6, 0$	$4, 1$
D	$5, 2$	$0, 3$

(a) Find all Nash equilibria of the stage game.

Solution. The unique rationalizable action profile is (M, R) and hence it is also the unique NE.

(b) According to the minmax folk theorem, what payoffs are possible to attain in a subgame perfect equilibrium when G is infinitely repeated and the common discount factor δ does to 1? (You are allowed to answer with a picture.)

Solution. The minmax folk theorem says that for all strictly individually rational (= strictly above the minmax payoffs) and feasible payoff vectors we can find large enough δ such tat there is a subgame perfect equilibrium with those payoffs.

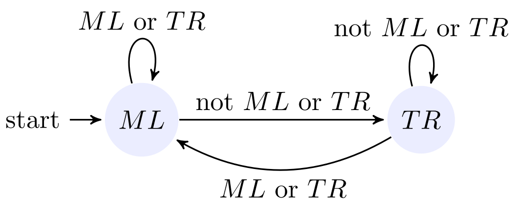
Minmax payoffs in G: $(4, -2)$.

Feasible payoffs in G: the convex hull of points $(1, -2)$, $(3, -2)$, $(6, 0)$, $(0, 3)$, and $(5, 2)$ ($(4, 1)$ is its interior point).

Hence all payoff vectors that are feasible and strictly larger than the minmax payoffs are all points in the convex hull of points $(4, -1.2)$, $(4, 2.2)$, $(5, 2)$, and $(6, 0)$, excluding the lower boundary points (though $(4, 1)$ is necessarily a SPE payoff because it is the stage NE). To get full points, one does not need to explicitly find the corner points: e.g. picture is enough.

(c) Draw an automaton that describes the following strategy profile for the infinitely many times repeated game: P1 plays M in the first period and if either (M, L) or (T, R) was played in the previous period, otherwise P1 plays T; P2 plays L in the first period and if either (M, L) or (T, R) was played in the previous period, otherwise P2 plays R.

Solution.



(d) Is the strategy profile in part (c) a subgame perfect equilibrium for some $\delta \in (0, 1)$?

Solution. We check for one-shot deviations in both automaton states for both players. To do that, we first calculate the values in each state:

$$V_1(ML) = 6$$
$$V_1(TR) = 3(1-\delta) + \delta 6$$
$$V_2(ML) = 0$$
$$V_2(TR) = -2(1-\delta)$$

ML, P1: no profitable deviation since both the one-shot payoff and the continuation value would be lower.

ML, P2: no profitable deviation if

$$0 \geq (1-\delta) + \delta(-2)(1-\delta) \iff \delta \geq 1/2.$$

TR, P1: the best possible deviation is to M. No profitable deviation if

$$3(1-\delta) + \delta 6 \geq 4(1-\delta) + \delta(3(1-\delta) + \delta 6) \iff \delta \geq 1/3.$$

TR, P2: no profitable deviation because L would give the same one-shot payoff and a lower continuation value.

We conclude that the suggested profile is a subgame perfect equilibrium for all $\delta \geq 1/2$.

4. Consider the following (fictional) situation. Denmark has one vacant slot for the Olympic games in cycling. There are two candidates who may get chosen and the final decision is based on a qualification race that takes place before the Olympics in the same season. Should the athletes time their best performance for the qualification race or for the Olympics?

You are allowed to combine sub questions but then you need to state clearly which sub questions you are answering together.

(a) Define a game that describes the situation.

Solution (example).

- Players: P1 and P2
- Actions and payoffs:

	Q	O
Q	$1/2\underline{v}, 1/2\underline{v}$	$p\underline{v}, (1-p)\overline{v}$
O	$(1-p)\overline{v}, p\underline{v}$	$1/2\overline{v}, 1/2\overline{v}$

where Q stands for timing the best performance for the qualification race and O stands for timing the best performance for the Olympics and $p > 0.5$ is the probability of winning the slot if timing for Q when the opponent times for O and the payoffs for entering the Olympics are $\overline{v}$ if timing for O and $\underline{v} \in (0, \overline{v})$ if timing for Q.

(b) Point out what assumptions you have made in part (a).

Solution (example). Assumptions:

- Players are symmetric.
- No gains (or losses) from balancing the best effort equally between the two competitions.
- Simultaneous moves.
- A higher chance to get selected if Q but higher payoff conditional on being selected if O.

(c) What would be a suitable solution concept to solve the game in part (a)? Argue why.

Solution (example). Can use NE because this is a simultaneous move game with complete information.

(d) Write down the equations that characterize a solution (this means that a strategy profile that satisfies all of them is a solution).

Solution (example). A mixed strategy profile $((\alpha_1, 1-\alpha_1), (\alpha_2, 1-\alpha_2))$ is a NE if

$$\alpha_1 \in \arg\max_{\alpha} \Big(\alpha\alpha_2 1/2\underline{v} + \alpha(1-\alpha_2)p\underline{v} + (1-\alpha)\alpha_2(1-p)\overline{v} + (1-\alpha)(1-\alpha_2)1/2\overline{v}\Big)$$
$$\alpha_2 \in \arg\max_{\alpha} \Big(\alpha\alpha_1 1/2\underline{v} + \alpha(1-\alpha_1)p\underline{v} + (1-\alpha)\alpha_1(1-p)\overline{v} + (1-\alpha)(1-\alpha_1)1/2\overline{v}\Big)$$

(e) Either solve the game OR discuss what you would expect to happen in the game (the latter means writing a few sentences where you describe the main tradeoff).

Solution (example). We solve the game under the assumption that $p = 1$. The game becomes

	Q	O
Q	$1/2\underline{v}, 1/2\underline{v}$	$\underline{v}, 0$
O	$0, \underline{v}$	$1/2\overline{v}, 1/2\overline{v}$

Now, if $\underline{v} > 1/2\overline{v}$, the unique NE is (Q, Q). If $\underline{v} < 1/2\overline{v}$, we have multiple equilibria: (Q, Q), (O, O), and symmetric mixing where both play Q w.p. $(\overline{v} - 2\underline{v})/(\overline{v} - \underline{v})$.

(f) Interpret your results (write a few sentences).

Solution (example). If getting into the Olympics is more important than having the best performance there (if $\underline{v} > 1/2\overline{v}$), then both athletes time their best performance for the qualifications, which is inefficient (similar to Prisoners' dilemma). If doing well in the Olympics is more important (if $\underline{v} < 1/2\overline{v}$), there are multiple equilibria, including the most efficient (O, O) equilibrium. However, the athletes may fail to coordinate to that equilibrium and end up in the inefficient (Q, Q) equilibrium (or in the mixed eqm) instead.